# Lecture 16

Tuesday, March 9, 2021 3:58 PM

* Prayer
* Spiritual thought
* Answering questions ...

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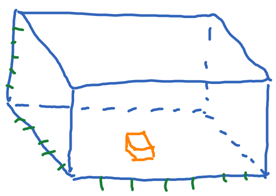
Triple integral



Box $B = [0,1] \times [0,2] \times [0,3]$.

Mass density $f(x,y,z) = yz + x$ $(g/cm^3)$.

What is the mass?

Partition the box into many small rectangular boxes (cells).

In each cell, the density is considered constant.

The mass of cell $ijk$ is approximately

$$f(x_i, y_j, z_k)\, \Delta V$$

Mass of the box $\approx \sum_{i,j,k} f(x_i, y_j, z_k)\, \Delta V$.

Definition:

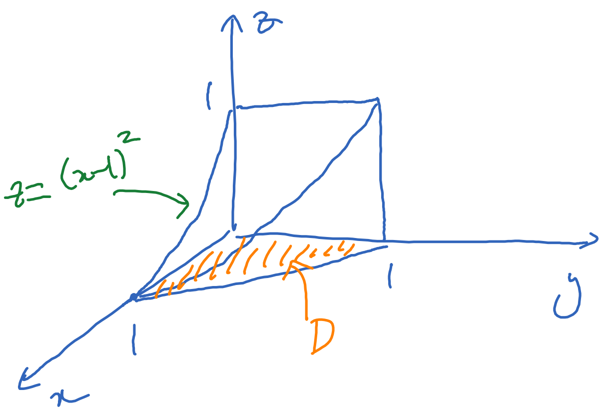
$$\iiint_B f(x,y,z)\, dV = \lim \sum_{i,j,k} f(x_i, y_j, z_k)\, \Delta V$$

How to compute this integral?

Convert it into an iterated integral.

$$\iiint_B f(x,y,z)\,dV = \int_0^1\int_0^2\int_0^3 (yz+x)\,dz\,dy\,dx.$$

If the solid is not a rectangular box, how do we compute its mass?




$$\iiint_E f(x,y,z)\,dV = ?$$

We can describe E as:

$$E = \{(x,y,z) : (x,y) \in D\text{ - triangle},\ 0 \le z \le (x-1)^2\}.$$

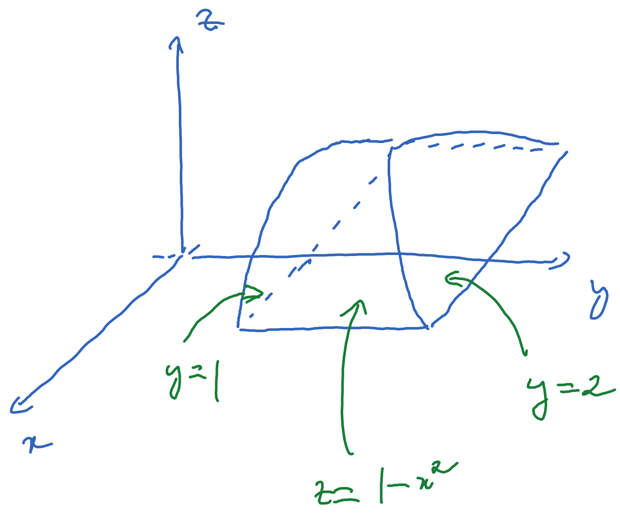
Then

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_0^{(x-1)^2} (yz+x)\,dz\,dA$$

D is described as $\{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$.

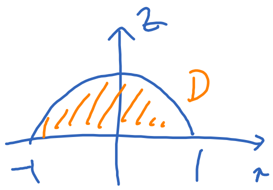
$$\iiint_E f(x,y,z)\,dV = \int_0^1\int_0^{1-x}\int_0^{(x-1)^2} (yz+x)\,dz\,dy\,dx.$$

Ex:




$$\iiint_E (yz+x)\,dV = ?$$

$E = \{(x,y,z) : 1 \le y \le 1-x^2, \ (x,z) \in D\}$.




Show Mathematica commands .....

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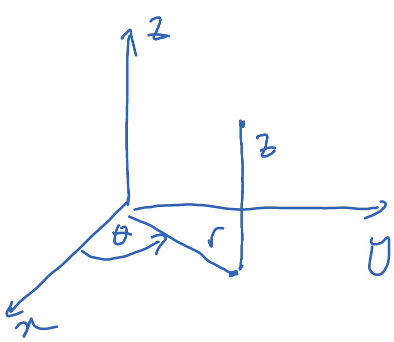
Cylindrical coordinate:




$E: \ g(x,y) \le z \le h(x,y)$

$(x,y) \in D$

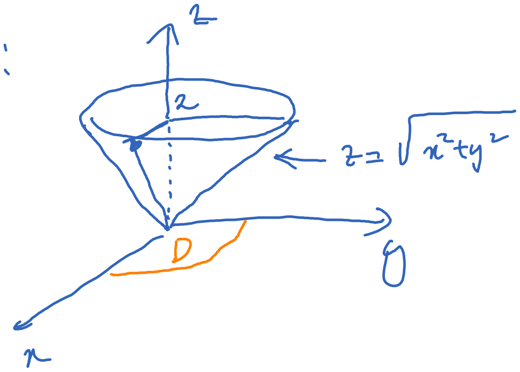
↑

polar region

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz\,dA$$

Then convert to polar coordinates from here.

Ex:




$f(x,y,z) = xz$

$E$ is the quarter cone.

$$\iiint_E f(x,y,z)\,dV = \,?$$

$$E: \quad (x,y) \in D, \quad \sqrt{x^2+y^2} \le z \le 2.$$

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## Change of variables

Recall:

$$\int_{[a,b]} f(x)\,dx$$

$x = g(u)$

$$\int_{[c,d]} f(g(u)) \underbrace{|g'(u)|}_{\text{"stretching factor"}} du$$

$[c,d]$ ← updated bounds

$$\iint_D f(x,y)\,dA$$

$x = g(u,v)$

$y = h(u,v)$

$$\iint_{D'} f(g(u,v), h(u,v)) \boxed{\;?\;} \,dA$$

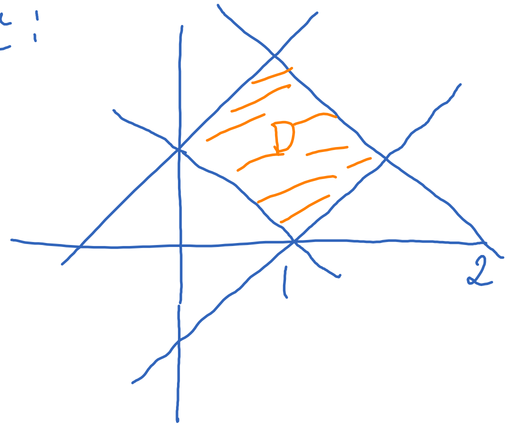
$D'$ ← updated bounds

$\boxed{?}$ ← stretching term

Stretching term $= \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right|$

$$= \left| \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \right|$$

Ex:




$D = \{(x,y) : 1 \le x+y \le 2, \; -1 \le x-y \le 1\}$

$$\iint_D x^2 \, dA = ?$$

$$\begin{cases} u = x+y \\ v = x-y \end{cases} \qquad D' = [1,2] \times [-1,1].$$

$$\rightsquigarrow \begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{u-v}{2} \end{cases}$$

$$J = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = |\det J| = \frac{1}{2}$$

$$\iint_D \cdots = \iint_{D'} \left(\frac{u+v}{2}\right)^2 \frac{1}{2} \, dA = \int_1^2 \int_{-1}^1 \frac{(u+v)^2}{2} \frac{1}{2} \, dv \, du = \cdots$$